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LETTER TO THE EDITOR

Lax equation, Lie algebra automorphism and nonlinear supersymmetric dynamical systems

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Abstract. Following an idea of Bogoyavlensky it is shown that a nonlinear supersymmetric integrable system can be constructed by using the automorphisms of supersymmetric Lie algebra.

Enlarging the class of integrable systems is one of the most important problems in the theory of nonlinear system [1]. Already there exists several concrete methods for the construction and solution of various types of nonlinear system [2]. Also, supersymmetric integrable systems have been studied by Mathieu [3], Kupershmidt [4], Manin [5] and others. All these analyses deal with the super generalizations of the usual integrable systems such as KP, KDV, sine-Gordon, etc. On the other hand supersymmetric dynamical systems (that is nonlinear systems without any space variation) have not received as much attention. Of late, an important idea was put forward by Bogoyavlensky [6], who showed that automorphism of Lie algebra [7] can be judiciously used to generate integrable systems of Toda lattice type. In this letter we show that by using supersymmetric Lie algebra and its usual automorphism it is possible to construct new integrable supersymmetric hierarchies. In the following we start with a short derivation of the basic result due to Bogoyavlensky and then utilize it in the case of supersymmetric Lie algebra.

Let us start from the equation [6]

$$\dot{L} = L\tau(M) - ML \tag{1}$$

where τ is an automorphism obeying the following conditions

$$\begin{aligned} \tau(M+N) &= \tau(M)+\tau(N) \\ \tau(MN) &= \tau(M)\tau(N) \end{aligned} \tag{2}$$

the order of the automorphism being N, i.e.,

$$\tau^N = 1. \tag{3}$$

Let us define

$$L_N = L\tau(L)\tau^2(L)\ldots\tau^{N-2}(L)\tau^{(N-1)}(L). \tag{4}$$

Now we find $(\tau^k(\dot{L}))$ given by

$$\begin{aligned}(\tau^k(\dot{L})) &= \tau^k(L\tau(M) - ML)\\ &= \tau^k(L\tau(M)) - \tau^k(ML)\\ &= \tau^k L\tau^k\tau(M) - \tau^k M\tau^k L\\ &= \tau^k(L)\tau^{k+1}(M) - \tau^k(M)\tau^k(L).\end{aligned} \tag{5}$$

This allows us to determine $\dot{L}_N$ as follows

$$\begin{aligned}\dot{L}_N &= \dot{L}\prod_{k=1}^{N-1}\tau^k(L) + L\tau(\dot{L})\prod_{k=2}^{N-1}\tau^k(L) + L\tau(L)\tau^2(\dot{L})\prod_{k=3}^{N-1}\tau^k(L) + \cdots\\ &\qquad + L\tau(L)\tau^2(L)\ldots\tau^{N-2}(L)\tau^{(N-1)}(\dot{L})\\ &= (L\tau(M) - ML)\prod_{k=1}^{N-1}\tau^k(L) + L(\tau(L)\tau^2(M) - \tau(M)\tau(L))\\ &\qquad \times\prod_{k=2}^{(N-1)}\tau^k(L) + L\tau(L)(\tau^2(L)\tau^3(M) - \tau^2(M)\tau^2(L))\prod_{k=3}^{N-1}\tau^k(L) + \cdots\\ &\qquad + L\tau(L)\tau^2(L)\ldots\tau^{N-2}(L)(\tau^{N-1}(L)\tau^N(M) - \tau^{N-1}(M)\tau^{N-1}(L))\\ &= -ML_N + \sum_{k=0}^{N-2}L\tau(L)\ldots\tau^k(L)\tau^{k+1}(M)\tau^{k+1}(L)\ldots\tau^{N-1}(L)\\ &\qquad - \sum_{k=1}^{N-1}L\tau(L)\ldots\tau^{k-1}(L)\tau^k(M)\tau^k(L)\ldots\tau^{N-1}(L) + L_N\tau^N(M)\\ &\Rightarrow \dot{L}_N = L_N\tau^N(M) - ML_N\end{aligned}$$

with the second and third terms cancelling out each other

$$\Rightarrow \dot{L}_N = L_N M - ML_N$$

since $\tau^N(M) = M$

$$\Rightarrow \dot{L}_N = [L_N, M]. \tag{6}$$

This is the form of Lax equation involving an automorphism of L itself which we will use in our subsequent analysis.

If the automorphism is of order two, i.e. $\sigma^2 = \mathbb{1}$, then the Lax equation (6) reduces to

$$\dot{L}_2 = [L_2, M] \tag{7}$$

where

$$L_2 = L\tau(L). \tag{8}$$

Equations (7) and (8) will be used, in particular, to derive our results.

The graded algebra $OSP(1,2)$ has three even (bosonic) generators, H, $J_{\pm}$ generating an $Sl(2)$ subalgebra, and two odd (fermionic) generators $V_{\pm}$ with the following commutation/anticommutation relations.

$$
\begin{aligned}
&[H, J_{\pm}] = \pm J_{\pm} \qquad [J_+, J_-] = 2H \\
&[H, V_{\pm}] = \pm\tfrac{1}{2} V_{\pm} \qquad [J_{\pm}, V_{\mp}] = V_{\pm} \qquad [J_{\pm}, V_{\pm}] = 0 \\
&\{V_{\pm}, V_{\pm}\} = \pm\tfrac{1}{2} J_{\pm} \qquad \{V_+, V_-\} = -\tfrac{1}{2} H.
\end{aligned}
\tag{9}
$$

All finite dimensional representations of $OSP(1,2)$ are completely reducible. The irreducible ones ρ_j are parametrized by an integer or half integer $j \epsilon N/2$, and have dimension $(4j+1)$. The fundamental representation has $j = \frac{1}{2}$. The generators can be represented as

$$
\begin{aligned}
&H = \tfrac{1}{2}(e_{11} - e_{33}) \qquad J_+ = e_{13} \qquad J_- = e_{31} \\
&V_+ = \tfrac{1}{2}(e_{12} + e_{23}) \qquad V_- = \tfrac{1}{2}(-e_{21} + e_{32})
\end{aligned}
\tag{10}
$$

where e_{ij} is a 3×3 matrix having 1 at the position of the ith row and jth column and zero everywhere else.

The $OSP(1,2)$ algebra admits the following decomposition

$$\alpha = \alpha^+ \oplus \alpha^- \tag{11}$$

where α^+ stands for the bosonic (even) part and α^- stands for the fermionic (odd) part of the algebra.

This means we can write

$$\alpha^+ = \{H, J_+, J_-\} \qquad \text{and} \qquad \alpha^- = \{V_+, V_-\} \tag{12}$$

with the following properties holding

$$[\alpha^+, \alpha^+] \subset \alpha^+ \qquad [\alpha^+, \alpha^-] \subset \alpha^- \qquad \{\alpha^-, \alpha^-\} \subset \alpha^+ \tag{13}$$

as may be readily checked from the communication relations (13). The above discussions shows that the algebra α admits an involutive automorphism τ,

$$\tau : \alpha \to \alpha \tag{14}$$

where

$$\tau^2 = \mathrm{id} \qquad \tau(x+y) = \tau(x) + \tau(y) \qquad [\tau(x), \tau(y)] = \tau([x, y]) \tag{15}$$

where

$$x, y \in \alpha$$

and the action of τ given by

$$\tau(\alpha^{\pm}) = \pm\alpha^{\pm}. \tag{16}$$

The Lax pair L, M can be constructed from the algebra $OSP(1,2)$ following the standard procedure. We write

$$\begin{aligned} L &= \mathrm{i}\lambda H + q_+J_+ + q_-J_- + r_+V_+ + r_-V_- \\ M &= A_0H + A_+J_+ + A_-J_- + B_+V_+ + B_-V_- \end{aligned} \tag{17}$$

where λ is the spectral parameter/eigenvalue, $q_\pm$, $r_\pm$ are the nonlinear fields with the q's bosonic and the r's fermionic, consistent with the supersymmetry requirements of the graded algebra under consideration.

In the same way, $A_\pm$ are the bosonic and $B_\pm$ the fermionic functions. The prescription (16) now allows us to write

$$\tau(L) = \mathrm{i}\lambda H + q_+J_+ + q_-J_- - r_+V_+ - r_-V_-. \tag{18}$$

These structures will be used in the following sections to derive nonlinear differential equations. From equation (8) we observe that L_2 is quadratic in the generators of the Lie algebra. To simplify such an expression we need the explicit form of these in terms of the basic matrices e_{ij} as given in equation (10). Whence we obtain,

$$\begin{aligned} L_2 = (q_+q_-)e_{11} + (q_-q_+)e_{33} - \left(\frac{\mathrm{i}\lambda}{2}r_+ + q_+r_-\right)\frac{1}{2}(e_{12}+e_{23}) + \left(\frac{\mathrm{i}\lambda}{2}r_- - q_-r_+\right)\frac{1}{2}(e_{32}-e_{21}) \\ - 1/4(r_+r_-)(e_{22}-e_{11}) - 1/4(r_-r_+)(e_{33}-e_{22}) \end{aligned} \tag{19}$$

where we have used the multiplication rule, $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Substituting now in equation (8), we at once get $B_- = B_+ = 0$ along with

$$\begin{aligned} \frac{1}{2}\left(\frac{\mathrm{i}\lambda}{2}\dot{r}_+ + (q_+r_-)'\right) &= \frac{1}{2}\left(\frac{\mathrm{i}\lambda}{2}r_- - q_-r_+\right)A_+ - \frac{1}{4}\left(\frac{\mathrm{i}\lambda}{2}r_+ + q_+r_-\right)A_0 \\ \frac{1}{2}\left(\frac{\mathrm{i}\lambda}{2}\dot{r}_- - (q_-r_+)'\right) &= \frac{1}{2}\left(\frac{\mathrm{i}\lambda}{2}r_- - q_-r_+\right)A_0 + \frac{1}{2}\left(\frac{\mathrm{i}\lambda}{2}r_+ + q_+r_-\right)A_-. \end{aligned} \tag{20}$$

We then expand $A_\pm$, A_0 as functions in the spectral parameter λ. For simplicity we set,

$$A_\pm = A_\pm^0 + \lambda A_\pm^1 + \lambda^2 A_\pm^2 \qquad A_0 = A_0^0 + \lambda A_0^1 + \lambda^2 A_0^2. \tag{21}$$

The unknown coefficients A_j^i's are determined in stages, whence we obtain;

$$\begin{aligned} &A_0^0(r_-r_+) = 2(\dot{q}_-r_-r_+ + q_-r_-\dot{r}_+) \\ &A_+^2 = q_+ \qquad A_-^2 = q_- \\ &A_0^2 = (4q_+q_-)^{1/2} \end{aligned} \tag{22}$$

so that finally the desired nonlinear equations turn out to be

$$\begin{aligned} &\dot{r}_+ = 4(q_+q_+)^{3/2}r_+ - 4q_-q_+^2r_- \\ &\dot{r}_- = -4q_-^2q_+r_- - 4(q_+q_-)^{3/2}r_- \\ &\dot{q}_+r_+ + 2q_+\dot{r}_- = 0 \\ &\dot{q}_-r_+ + 2q_-\dot{r}_+ = 0. \end{aligned} \tag{23}$$

In all the above expressions the dot $(\cdot)$ over the functions denotes time derivative.

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